Algebraic Geometry Final Exam

April 19 2024

This exam is of **50 marks** and is **3 hours long**. Please **read all the questions carefully**. Please feel free to use whatever theorems you have learned in class after stating them clearly. You may consult the books of Miles Reid - '**Undergraduate Algebraic Geometry**' and W. Fulton - '**Algebraic Curves**'.

1. Let p be an **odd prime** number.

    1. Under what conditions on p does the equation
    $$C_p : X^2 + Y^2 = p$$
    have a rational solution? (3)
    2. Construct a birational map from $\mathbb{P}^1 \to C_{17}$ and use that to find all rational solutions. (2)
    3. Under what conditions on p does the equation
    $$D_p : X^2 - Y^2 = p$$
    have a rational solution? (3)
    4. Construct a birational map from $\mathbb{P}^1 \to D_{11}$ and use that to find all rational solutions. (2)

2. Let J be the ideal $J = (XY, YZ, XZ) \subset k[X, Y, Z]$.

    1. What is $V(J)$? (2)
    2. Is $V(J)$ irreducible? (2)
    3. Is $J = I(V(J))$? (2)
    4. What is $Rad(J)$? (2)
    5. Prove that J cannot be generated by 2 elements. (2)

3. Let $C \subset \mathbb{P}^3$ be given by $Q_1 \cap Q_2 \cap Q_3$ where

$$Q_1 : XZ = Y^2 \qquad Q_2 : XW = YZ \qquad Q_3 : YW = Z^2$$

   1. Show that $C \neq Q_i \cap Q_j$ for any two i and j. (4)

   2. Show that C is parameterised by (3)

$$\mathbb{P}^1 \longrightarrow C$$

$$[U, V] \longrightarrow [U^3, U^2V, UV^2, V^3]$$

   3. What is the inverse map? (3)

4. Let $C = V(F)$ be an **affine plane curve**.

   1. Show that C has finitely many multiple points. (2)

   2. For $F(X, Y) = Y^2 - X^7$ and $C = V(F)$ find F' where F' is the equation of the curve birational to C as in Section 7.2 of Fulton. (3)

   3. Let $C' = V(F')$. Is it non-singular? If not, find a curve C'' birational to C' which is non-singular. (3)

   4. Describe the map $C'' \longrightarrow C$. (2)

5. Let C be a **cubic plane curve**.

   1. How many multiple points can C have. (2)

   2. What are the possible genera (genus) that C can have? (3)

   3. Under what conditions on C do there exist two distinct points P and Q such that $P \sim Q$? Justify your answer. (5)